

## Exercise 10

The integration formula

$$\int_0^{\infty} \frac{dx}{[(x^2 - a)^2 + 1]^2} = \frac{\pi}{8\sqrt{2}A^3} [(2a^2 + 3)\sqrt{A + a} + a\sqrt{A - a}],$$

where  $a$  is any real number and  $A = \sqrt{a^2 + 1}$ , arises in the theory of case-hardening of steel by means of radio-frequency heating.\* Follow the steps below to derive it.

- (a) Point out why the four zeros of the polynomial

$$q(z) = (z^2 - a)^2 + 1$$

are the square roots of the numbers  $a \pm i$ . Then, using the fact that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A + a} + i\sqrt{A - a})$$

and  $-z_0$  are the square roots of  $a + i$  (Exercise 5, Sec. 10), verify that  $\pm \bar{z}_0$  are the square roots of  $a - i$  and hence that  $z_0$  and  $-\bar{z}_0$  are the only zeros of  $q(z)$  in the upper half plane  $\text{Im } z \geq 0$ .

- (b) Using the method derived in Exercise 7, Sec. 76, and keeping in mind that  $z_0^2 = a + i$  for purposes of simplification, show that the point  $z_0$  in part (a) is a pole of order 2 of the function  $f(z) = 1/[q(z)]^2$  and that the residue  $B_1$  at  $z_0$  can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a - i(2a^2 + 3)}{16A^2 z_0}.$$

After observing that  $q'(-\bar{z}) = -\overline{q'(z)}$  and  $q''(-\bar{z}) = \overline{q''(z)}$ , use the same method to show that the point  $-\bar{z}_0$  in part (a) is also a pole of order 2 of the function  $f(z)$ , with residue

$$B_2 = \overline{\left\{ \frac{q''(z_0)}{[q'(z_0)]^3} \right\}} = -\bar{B}_1.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2 i} \text{Im} \left[ \frac{-a + i(2a^2 + 3)}{z_0} \right]$$

for the sum of the residues.

- (c) Refer to part (a) and show that  $|q(z)| \geq (R - |z_0|)^4$  if  $|z| = R$ , where  $R > |z_0|$ . Then, with the aid of the final result in part (b), complete the derivation of the integration formula.

### Solution

\*See pp. 359–364 of the book by Brown, Hoyer, and Bierwirth that is listed in Appendix 1.

The integrand is an even function of  $x$ , so the interval of integration can be extended to  $(-\infty, \infty)$  as long as the integral is divided by 2.

$$\int_0^{\infty} \frac{dx}{[(x^2 - a)^2 + 1]^2} = \int_{-\infty}^{\infty} \frac{dx}{2[(x^2 - a)^2 + 1]^2}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{1}{2[(z^2 - a)^2 + 1]^2},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$2[(z^2 - a)^2 + 1]^2 = 0$$

$$(z^2 - a)^2 + 1 = 0$$

$$z^2 - a = \pm i$$

$$z^2 = a \pm i \rightarrow \begin{cases} z_1 = \sqrt{a+i} = \frac{1}{\sqrt{2}} \left( \sqrt{\sqrt{a^2+1}+a} + i\sqrt{\sqrt{a^2+1}-a} \right) \\ z_2 = \sqrt{a-i} = \frac{1}{\sqrt{2}} \left( \sqrt{\sqrt{a^2+1}+a} - i\sqrt{\sqrt{a^2+1}-a} \right) \\ z_3 = -\sqrt{a+i} = \frac{1}{\sqrt{2}} \left( -\sqrt{\sqrt{a^2+1}+a} - i\sqrt{\sqrt{a^2+1}-a} \right) \\ z_4 = -\sqrt{a-i} = \frac{1}{\sqrt{2}} \left( -\sqrt{\sqrt{a^2+1}+a} + i\sqrt{\sqrt{a^2+1}-a} \right) \end{cases}$$

Proof for these alternative representations of  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  will be given at the end. The singular points of interest to us are the ones that lie within the closed contour,  $z = z_1$  and  $z = z_4$ .

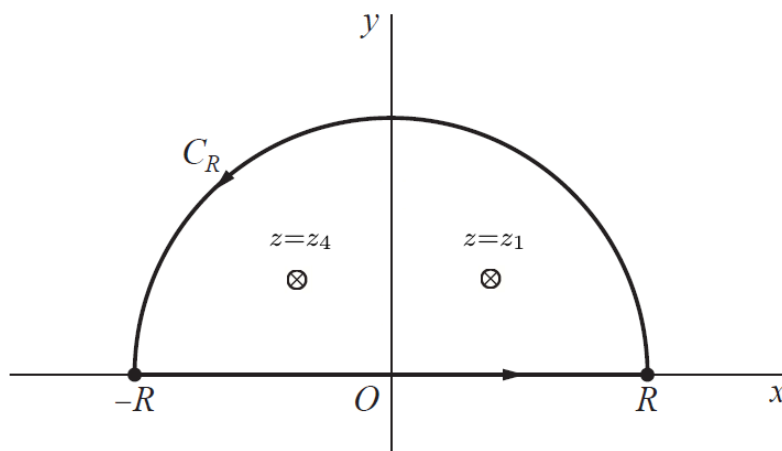


Figure 1: This is Fig. 93 with the singularities at  $z = z_1$  and  $z = z_4$  marked.

According to Cauchy's residue theorem, the integral of  $1/\{2[(z^2 - a)^2 + 1]^2\}$  around the closed contour is equal to  $2\pi i$  times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{2[(z^2 - a)^2 + 1]^2} = 2\pi i \left\{ \operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} + \operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} \right\}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{dz}{2[(z^2 - a)^2 + 1]^2} + \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} = 2\pi i \left\{ \operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} + \operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} \right\}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z &= r, & r &= -R \rightarrow r = R \\ C_R: \quad z &= Re^{i\theta}, & \theta &= 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^R \frac{dr}{2[(r^2 - a)^2 + 1]^2} + \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} = 2\pi i \left\{ \operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} + \operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} \right\}.$$

Take the limit now as  $R \rightarrow \infty$ . The integral over  $C_R$  consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{dr}{2[(r^2 - a)^2 + 1]^2} = 2\pi i \left\{ \operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} + \operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} \right\}.$$

The denominator can be written as  $2[(z^2 - a)^2 + 1]^2 = 2[q(z)]^2$ . Because  $q(z_1) = 0$  and  $q'(z_1) \neq 0$ ,  $q(z)$  is said to have a zero of order 1 at  $z = z_1$  and can be written as

$$q(z) = (z - z_1)\phi_1(z), \quad (1)$$

where  $\phi_1(z_1) \neq 0$ . Then

$$\begin{aligned} \operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} &= \operatorname{Res}_{z=z_1} \frac{1}{2(z - z_1)^2 [\phi_1(z)]^2} \\ &= \operatorname{Res}_{z=z_1} \frac{1/\{2[\phi_1(z)]^2\}}{(z - z_1)^2}. \end{aligned}$$

We see from this form that  $z = z_1$  is a pole of order 2; the residue can then be calculated by

$$\begin{aligned} &= \frac{d}{dz} \left\{ \frac{1}{2[\phi_1(z)]^2} \right\} \Big|_{z=z_1} \\ &= -\frac{\phi_1'(z)}{[\phi_1(z)]^3} \Big|_{z=z_1} \\ &= -\frac{\phi_1'(z_1)}{[\phi_1(z_1)]^3}. \end{aligned}$$

Differentiate both sides of equation (1) with respect to  $z$  twice in order to write the previous formula in terms of  $q(z)$ .

$$\begin{aligned} q'(z) &= \phi_1(z) + (z - z_1)\phi_1'(z) & \Rightarrow & \quad q'(z_1) = \phi_1(z_1) \\ q''(z) &= 2\phi_1'(z) + (z - z_1)\phi_1''(z) & \Rightarrow & \quad q''(z_1) = 2\phi_1'(z_1) \rightarrow \phi_1'(z_1) = \frac{1}{2}q''(z_1) \end{aligned}$$

Consequently,

$$\begin{aligned}\operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} &= -\frac{\frac{1}{2}q''(z_1)}{[q'(z_1)]^3} \\ &= -\frac{1}{2} \frac{[(z^2 - a)^2 + 1]''}{\{[(z^2 - a)^2 + 1]'\}^3} \Big|_{z=z_1} \\ &= -\frac{1}{2} \frac{4(3z_1^2 - a)}{[4z_1(z_1^2 - a)]^3}.\end{aligned}$$

Note that  $z_1^2 = a + i$ .

$$\begin{aligned}&= -\frac{(2a + 3i)}{32z_1^3(i)^3} \\ &= -\frac{2a + 3i}{32z_1(a + i)(-i)} \\ &= \frac{2a + 3i}{32z_1(ia - 1)} \cdot \frac{-ia - 1}{-ia - 1} \\ &= \frac{a - i(2a^2 + 3)}{32(a^2 + 1)z_1}\end{aligned}$$

Similarly, because  $q(z_4) = 0$  and  $q'(z_4) \neq 0$ ,  $q(z)$  is said to have a zero of order 1 at  $z = z_4$  and can be written as

$$q(z) = (z - z_4)\phi_2(z), \quad (2)$$

where  $\phi_2(z_4) \neq 0$ . Then

$$\begin{aligned}\operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} &= \operatorname{Res}_{z=z_4} \frac{1}{2(z - z_4)^2[\phi_2(z)]^2} \\ &= \operatorname{Res}_{z=z_4} \frac{1/\{2[\phi_2(z)]^2\}}{(z - z_4)^2}.\end{aligned}$$

We see from this form that  $z = z_4$  is a pole of order 2; the residue can then be calculated by

$$\begin{aligned}&= \frac{d}{dz} \left\{ \frac{1}{2[\phi_2(z)]^2} \right\} \Big|_{z=z_4} \\ &= -\frac{\phi_2'(z)}{[\phi_2(z)]^3} \Big|_{z=z_4} \\ &= -\frac{\phi_2'(z_4)}{[\phi_2(z_4)]^3}.\end{aligned}$$

Differentiate both sides of equation (2) with respect to  $z$  twice in order to write the previous formula in terms of  $q(z)$ .

$$\begin{aligned}q'(z) &= \phi_2(z) + (z - z_4)\phi_2'(z) &\Rightarrow q'(z_4) &= \phi_2(z_4) \\ q''(z) &= 2\phi_2'(z) + (z - z_4)\phi_2''(z) &\Rightarrow q''(z_4) &= 2\phi_2'(z_4) \quad \rightarrow \quad \phi_2'(z_4) = \frac{1}{2}q''(z_4)\end{aligned}$$

Consequently,

$$\begin{aligned} \operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} &= -\frac{\frac{1}{2}q''(z_4)}{[q'(z_4)]^3} \\ &= -\frac{1}{2} \frac{[(z^2 - a)^2 + 1]''}{\{[(z^2 - a)^2 + 1]'\}^3} \Big|_{z=z_4} \\ &= -\frac{1}{2} \frac{4(3z_4^2 - a)}{[4z_4(z_4^2 - a)]^3}. \end{aligned}$$

Note that  $z_4^2 = a - i$ .

$$\begin{aligned} &= -\frac{(2a - 3i)}{32z_4^3(-i)^3} \\ &= \frac{2a - 3i}{32z_4(a - i)(-i)} \\ &= \frac{2a - 3i}{32z_4(-ia - 1)} \cdot \frac{ia - 1}{ia - 1} \\ &= \frac{a + i(2a^2 + 3)}{32(a^2 + 1)z_4} \end{aligned}$$

Now that the residues are known, the integral can be evaluated.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dr}{2[(r^2 - a)^2 + 1]^2} &= 2\pi i \left\{ \operatorname{Res}_{z=z_1} \frac{1}{2[(z^2 - a)^2 + 1]^2} + \operatorname{Res}_{z=z_4} \frac{1}{2[(z^2 - a)^2 + 1]^2} \right\} \\ &= 2\pi i \left[ \frac{a - i(2a^2 + 3)}{32(a^2 + 1)z_1} + \frac{a + i(2a^2 + 3)}{32(a^2 + 1)z_4} \right] \\ &= 2\pi i \left[ \frac{az_4 - iz_4(2a^2 + 3) + az_1 + iz_1(2a^2 + 3)}{32(a^2 + 1)z_1z_4} \right] \\ &= 2\pi i \left[ \frac{a(z_1 + z_4) + i(2a^2 + 3)(z_1 - z_4)}{32(a^2 + 1)z_1z_4} \right] \\ &= 2\pi i \left[ \frac{a \cdot \frac{2i}{\sqrt{2}} \sqrt{\sqrt{a^2 + 1} - a} + i(2a^2 + 3) \cdot \frac{2}{\sqrt{2}} \sqrt{\sqrt{a^2 + 1} + a}}{32(a^2 + 1)(\sqrt{a + i})(-\sqrt{a - i})} \right] \\ &= 2\pi i \left[ \frac{(-i) \frac{a\sqrt{\sqrt{a^2 + 1} - a} + (2a^2 + 3)\sqrt{\sqrt{a^2 + 1} + a}}{16\sqrt{2}(a^2 + 1)^{3/2}}}{16\sqrt{2}(a^2 + 1)^{3/2}} \right] \\ &= \frac{\pi}{8\sqrt{2}(a^2 + 1)^{3/2}} \left[ (2a^2 + 3)\sqrt{\sqrt{a^2 + 1} + a} + a\sqrt{\sqrt{a^2 + 1} - a} \right] \end{aligned}$$

Therefore, changing the dummy integration variable to  $x$ ,

$$\boxed{\int_0^{\infty} \frac{dx}{[(x^2 - a)^2 + 1]^2} = \frac{\pi}{8\sqrt{2}(a^2 + 1)^{3/2}} \left[ (2a^2 + 3)\sqrt{\sqrt{a^2 + 1} + a} + a\sqrt{\sqrt{a^2 + 1} - a} \right].}$$

The Alternative Representations of  $z_1, z_2, z_3,$  and  $z_4$ 

Start by writing the polar form of  $a \pm i$ .

$$\begin{aligned}\sqrt{a \pm i} &= \sqrt{\sqrt{a^2 + 1} \exp \left[ i \tan^{-1} \left( \pm \frac{1}{a} \right) \right]} \\ &= \sqrt[4]{a^2 + 1} \exp \left[ \pm \frac{i}{2} \tan^{-1} \left( \frac{1}{a} \right) \right]\end{aligned}$$

Use Euler's formula.

$$= \sqrt[4]{a^2 + 1} \left\{ \cos \left[ \frac{1}{2} \tan^{-1} \left( \frac{1}{a} \right) \right] \pm i \sin \left[ \frac{1}{2} \tan^{-1} \left( \frac{1}{a} \right) \right] \right\}$$

Apply the half-angle formulas for sine and cosine.

$$= \sqrt[4]{a^2 + 1} \left\{ \sqrt{\frac{1 + \cos \left[ \tan^{-1} \left( \frac{1}{a} \right) \right]}{2}} \pm i \sqrt{\frac{1 - \cos \left[ \tan^{-1} \left( \frac{1}{a} \right) \right]}{2}} \right\}$$

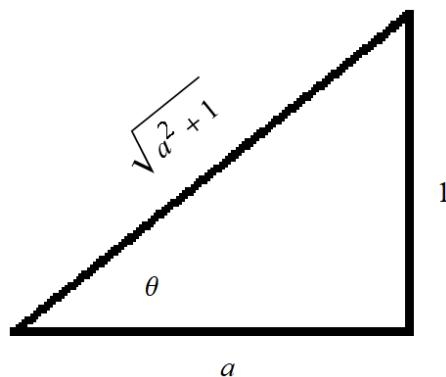


Figure 2: Draw the implied right triangle to determine the cosine of  $\theta = \tan^{-1} \left( \frac{1}{a} \right)$ .

$$\begin{aligned}\sqrt{a \pm i} &= \frac{\sqrt[4]{a^2 + 1}}{\sqrt{2}} \left( \sqrt{1 + \frac{a}{\sqrt{a^2 + 1}}} \pm i \sqrt{1 - \frac{a}{\sqrt{a^2 + 1}}} \right) \\ &= \frac{1}{\sqrt{2}} \left( \sqrt{\sqrt{a^2 + 1} + a} \pm i \sqrt{\sqrt{a^2 + 1} - a} \right)\end{aligned}$$

Therefore,

$$\begin{aligned}z_1 = \sqrt{a + i} &= \frac{1}{\sqrt{2}} \left( \sqrt{\sqrt{a^2 + 1} + a} + i \sqrt{\sqrt{a^2 + 1} - a} \right) \\ z_2 = \sqrt{a - i} &= \frac{1}{\sqrt{2}} \left( \sqrt{\sqrt{a^2 + 1} + a} - i \sqrt{\sqrt{a^2 + 1} - a} \right).\end{aligned}$$

The representations for  $z_3$  and  $z_4$  follow because  $z_3 = -z_1$  and  $z_4 = -z_2$ .

The Integral Over  $C_R$ 

Our aim here is to show that the integral over  $C_R$  tends to zero in the limit as  $R \rightarrow \infty$ . The parameterization of the semicircular arc in Fig. 93 is  $z = Re^{i\theta}$ , where  $\theta$  goes from 0 to  $\pi$ .

$$\begin{aligned} \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} &= \int_0^\pi \frac{Rie^{i\theta} d\theta}{2\{[(Re^{i\theta})^2 - a]^2 + 1\}^2} \\ &= \int_0^\pi \frac{Rie^{i\theta}}{[(R^2e^{i2\theta} - a)^2 + 1]^2} \frac{d\theta}{2} \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} \right| &= \left| \int_0^\pi \frac{Rie^{i\theta}}{[(R^2e^{i2\theta} - a)^2 + 1]^2} \frac{d\theta}{2} \right| \\ &\leq \int_0^\pi \left| \frac{Rie^{i\theta}}{[(R^2e^{i2\theta} - a)^2 + 1]^2} \right| \frac{d\theta}{2} \\ &= \int_0^\pi \frac{|Rie^{i\theta}|}{|[(R^2e^{i2\theta} - a)^2 + 1]^2|} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{|(R^2e^{i2\theta} - a)^2 + 1|^2} \frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R}{[|(R^2e^{i2\theta} - a)^2| - |1|]^2} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{(|R^2e^{i2\theta} - a|^2 - 1)^2} \frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R}{[(|R^2e^{i2\theta}| - |a|)^2 - 1]^2} \frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{[(R^2 - a)^2 - 1]^2} \frac{d\theta}{2} \\ &= \frac{\pi}{2} \frac{R}{[(R^2 - a)^2 - 1]^2} \end{aligned}$$

Now take the limit of both sides as  $R \rightarrow \infty$ .

$$\begin{aligned} \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} \right| &\leq \lim_{R \rightarrow \infty} \frac{\pi}{2} \frac{R}{[(R^2 - a)^2 - 1]^2} \\ &= \lim_{R \rightarrow \infty} \frac{\pi}{2} \frac{R}{\left[ R^4 \left( 1 - \frac{a}{R^2} \right)^2 - 1 \right]^2} \\ &= \lim_{R \rightarrow \infty} \frac{\pi}{2R^7} \frac{1}{\left[ \left( 1 - \frac{a}{R^2} \right)^2 - \frac{1}{R^4} \right]^2} \end{aligned}$$

The limit on the right side is zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} \right| \leq 0$$

The magnitude of a number cannot be negative, and only zero has zero magnitude. Therefore,

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} \right| = 0 \quad \rightarrow \quad \lim_{R \rightarrow \infty} \int_{C_R} \frac{dz}{2[(z^2 - a)^2 + 1]^2} = 0.$$